ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 340 (2008) 197–208
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A note on the extension of *BV* functions in metric measure spaces

Annalisa Baldi *, Francescopaolo Montefalcone [1]

Dipartimento di Matematica, Università degli Studi di Bologna, Piazza di P.ta S. Donato, 5, 40126 Bologna, Italy

Received 25 January 2007
Available online 25 July 2007
Submitted by Cristian Gutierrez

Abstract

In this note, a classical extension result for *BV* functions due to Yu.D. Burago and V.G. Maz'ja [Yu.D. Burago, V.G. Maz'ja, Potential Theory and Function Theory for Irregular Regions, Seminars in Math., vol. 3, V.A. Steklov Math. Inst., Leningrad, 1969 (translated from Russian)] is generalized to the abstract setting of doubling metric measure spaces endowed with a differentiation structure called *D*-structure.

Keywords: *BV*; Perimeter; Metric space; *D*-structure; Doubling condition

1. Introduction

In the last few years, there has been an increasing interest in the analysis in metric spaces and, in particular, in the study of function spaces as Sobolev or *BV* spaces defined on metric measure spaces supporting Poincaré inequalities. Motivating examples come from various subject such as Carnot–Carathéodory geometries, hypoelliptic PDE's, fractal geometries, singular Riemannian manifolds, analysis on graphs, etc. We shall refer the reader, for instance, to [1,3,4, 12,17,18,20,24,25,27,29], for a general setting. For more specific classes like Carnot–Carathéodory spaces or metric induced by doubling metric measures we mention [6–8,11,13,15,16,19,23,32], but of course this list is not complete. Nevertheless, despite the progress in the understanding of the Sobolev space theory in the general setting of metric spaces, the development of Geometric Measure Theory is far from being exhaustive. In this note we would like to give a little contribution in this direction.

More precisely, we are interested in the problem of the continuation to the whole space of a metric *BV* function defined on an open subset and, particularly, in the geometric characterization of those subsets of the metric space that allow this continuation property. In the Euclidean setting a sharp characterization of domains with such a continuation property for *BV* functions was proved by Yu.D. Burago and V.G. Maz'ja [10]; see also §6.3 in [28].

* Corresponding author.
E-mail addresses: baldi@dm.unibo.it (A. Baldi), montefal@dm.unibo.it (F. Montefalcone).
[1] Investigation supported by University of Bologna, funds for selected research topics and by GNAMPA of INdAM, Italy.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.07.039

The most interesting feature of this characterization is provided by the fact that the condition itself is not formulated in terms of differential properties of the boundary, but instead in purely geometric terms that have a natural counterpart in any metric space where a *BV* function theory can be carried out.

For a precise definition of *BV* functions and their properties we refer the reader to Section 2 where we shall follow the axiomatic approach introduced by Troyanov in [20] for the theory of Sobolev spaces, and then used by Ambrosio in [2] to investigate some fine properties of *BV* functions in this abstract setting (generalizing some previous results by Miranda [29]).

As explicitly mentioned by Grigor'yan in the paper [22], a beautiful feature of the mathematical work of Maz'ya is that the methods of proof of many of his results does not use specific properties of the Euclidean space, and therefore, can easily be adapted even in any Riemannian manifold. The same remark fits very well to the aforementioned paper by Burago and Maz'ja [10], whose arguments make a systematic use of isoperimetric inequalities and coarea formula. Similar tools are also available in our abstract metric setting; see [1,3,29].

Let now (X, d) denote a doubling metric space. Our first result is given in Theorem 3.3 where we shall state the equivalence between the existence of an extension of a *BV* function defined in an open set $\Omega \subset X$ and the validity of the following "isoperimetric-type inequality":

$$\tau(E, \Omega) \leqslant K P(E, \Omega) \quad \text{for all Borel sets } E \subset \Omega, \tag{1}$$

where the constant K is independent of E. Here $P(E, \Omega)$ is the perimeter of the set E in Ω and the set function $\tau(\cdot, \Omega): E \to \tau(E, \Omega)$ is defined by

$$\tau(E, \Omega) := \inf_{B \cap \Omega = E} P\big(B, \Omega^c\big)$$

for all Borel sets $B \subset X$. A local version of this extension result will be given in Theorem 3.8. A useful reformulation of Theorem 3.3 is stated in Corollary 3.9 which says that, given an open set $\Omega \subset X$, there exists a bounded linear operator $\mathcal{E}_\Omega : BV(\Omega) \to BV(X)$ if and only if Ω satisfies the isoperimetric condition (1) for some constant K.

We would like to remark that, unlike the Euclidean case, in our metric setting a global isoperimetric inequality does not make sense in general. Hence, some of our proofs follow different lines with respect to the corresponding results proved in [10]. In particular, throughout the proof of Theorem 3.8, we shall overcome the lack of a global isoperimetric inequality by means of a relative isoperimetric inequality together with the reverse doubling condition (see Proposition 2.8).

After this paper was written, similar extensions results for Sobolev functions in metric measure spaces have been obtained in [9,26,34]. In these works the extension property is connected with a regularity property of the extension domains Ω formulated in terms of volume. More precisely, a set $\Omega \subset X$ is said to satisfy the *measure density condition* if there is a constant $C > 0$ such that

$$\mu\big(\Omega \cap B(x, r)\big) \geqslant C\mu\big(B(x, r)\big) \tag{2}$$

for all balls $B(x, r)$ with $x \in \overline{\Omega}$ and $0 < r \leqslant 1$.

In those papers, the authors show the relation between the measure density condition for a bounded open set Ω and the existence of an extension operator $\mathcal{E} : M^{1,p}(\Omega) \to M^{1,p}(X)$ for $1 \leqslant p < \infty$ (here $M^{1,p}(X)$ denotes the class of Hajłasz–Sobolev spaces; see [24]). For the case $1 \leqslant p < \infty$ there are similar results for Newtonian spaces $N^{1,p}(X)$ defined in [33].

On the other hand, we stress again that, in our setting, the geometric property of the extension domain is formulated in terms of the perimeter measure instead of the volume one. At present, it is not completely understood the relationship between the measure density condition (2) and the isoperimetric-type property (1) and, in our opinion, this is an interesting problem, even in the Euclidean case.

Finally, we should also mention that, in the case of Carnot–Carathéodory spaces or homogeneous groups, results of this type have been proved for Sobolev spaces defined in special classes of domains such as NTA domains (i.e. non-tangential accessible domains), (ϵ, δ)-domains, also known in literature as uniform domains, or John domains. Some of these characterizations can be found in [13,14,21,30,31].

2. Some preliminary results

Let (X, d) be a complete metric space. If $r > 0$, let $B_r(x)$ denote the open ball $\{y \in X\colon d(x, y) < r\}$ and let $\mathcal{B}(X)$ denote the Borel σ-algebra of X.

We assume that (X,d) is endowed with a doubling measure μ, i.e. a σ-additive set function $\mu:\mathcal{B}(X)\to[0,+\infty]$, finite on bounded subsets of X and satisfying

$$\mu\big(B_{2r}(x)\big)\leqslant C_D\,\mu\big(B_r(x)\big)\quad \forall x\in X,\ r>0, \tag{3}$$

for some positive constant C_D. We remind that the doubling property implies the following density lower bound

$$\frac{\mu(B_\rho(x))}{\mu(B_R(y))}\geqslant c\left(\frac{\rho}{R}\right)^s\quad \forall 0<\rho\leqslant R<\infty,\ x\in B_R(y), \tag{4}$$

for some constant c depending on s and C_D, where s satisfies $s\geqslant \log_2 C_D$; see [2]. In a sense, the number s defines a "dimension" of the ambient space which is called the *homogeneous dimension* of X.

We should also notice that, since the balls are totally bounded, then closed balls are compact. This implies that the notion of doubling metric space is somehow finite-dimensional.

Along the lines of [2,20], we shall equip the metric space X with a differentiation structure, the so-called *D-structure*. To this end, we introduce below the axioms that the set of *pseudo-gradients* of a locally Lipschitz function must satisfy. More precisely, if $u\in \mathrm{Lip}_{\mathrm{loc}}(X)$ (i.e. the vector space of real-valued Lipschitz functions on bounded subsets of X) let $D[u]$ denote the set of non-negative Borel functions satisfying the following six axioms:

(A0) **Axiom 0.** (Non-triviality) $0\in D[u]$ for any constant function u.
(A1) **Axiom 1.** (Upper linearity) If $g_1\in D[u_1]$, $g_2\in D[u_2]$ and $g\geqslant|\alpha|g_1+|\beta|g_2$ μ-a.e., then $g\in D[\alpha u_1+\beta u_2]$.
(A2) **Axiom 2.** (Leibniz rule) If $g\in D[u]$ then $\sup\varphi g+\mathrm{Lip}(\varphi)|u|$ belongs to $D[\varphi u]$ whenever φ is a bounded Lipschitz function.
(A3) **Axiom 3.** (Lattice property) If $g_1\in D[u_1]$, $g_2\in D[u_2]$ and $u=\min\{u_1,u_2\}$, then $\max\{g_1,g_2\}\in D[u]$.
(A4) **Axiom 4.** (Locality) If $A\subset X$ is an open set and $g\in D[u]$, then the function

$$g'=\begin{cases} g & \text{on } X\setminus A,\\ h & \text{on } A\end{cases}$$

belongs to $D[u]$ whenever $h\in D[v]$ and $u\equiv v$ on A.
(A5) **Axiom 5.** (Weak Poincaré inequality) For any $g\in D[u]$ and any ball $B_r(x)$ we have

$$\int_{B_r(x)}\big|u(y)-u_{x,r}\big|\,d\mu(y)\leqslant C_P r\int_{B_{\lambda r}(x)} g(y)\,d\mu(y) \tag{5}$$

where $u_{x,r}$ denotes the mean value of u in $B_r(x)$, i.e.

$$u_{x,r}:=\frac{1}{\mu(B_r(x))}\int_{B_r(x)}u(y)\,d\mu(y).$$

Definition 2.1. The set valued map $u\to D[u]$ is called a *D-structure* if the axioms (A0)–(A4) hold. We say that the D-structure satisfies the *weak* $(1,1)$-*Poincaré inequality* if the axiom (A5) holds with $\lambda>1$. Finally, we say that the D-structure satisfies the $(1,1)$-*Poincaré inequality* if (5) holds with $\lambda=1$.

As noticed in [2], condition (5) in the last axiom (A5) could also be replaced by the following weaker one:

$$\min_{m\in\mathbb{R}}\int_{B_r(x)}\big|u(y)-m\big|\,d\mu(y)\leqslant Cr\int_{B_{\lambda r(x)}} g(y)\,d\mu(y). \tag{6}$$

Actually, it can be proved that (6) implies (5) with $C_P=2C$ (see [2]).

By using D-structures and a quite standard relaxation procedure, we shall define, along the lines of [2,29], the perimeter measure, the class of sets of finite perimeter and the class of metric BV functions.

Definition 2.2. Let $\Omega\subset X$ be open. Then, for any $u\in L^1_{\mathrm{loc}}(\Omega)$ we shall set

$$|Du|(\Omega):=\inf\left\{\liminf_{h\to+\infty}\int_\Omega g_h\,d\mu:\ (u_h)\subset \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ u_h\xrightarrow{L^1_{\mathrm{loc}}(\Omega)}u,\ g_h\in D[u_h]\right\}.$$

The quantity $|Du|(\Omega)$ is called the *variation of u in* Ω and $BV(\Omega)$ denotes the space of all functions $u \in L^1(\Omega)$ such that $|Du|(\Omega) < +\infty$. The vector space $BV(\Omega)$ is naturally equipped with the norm $\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|(\Omega)$. Furthermore if $E \in \mathcal{B}(X)$, the *perimeter of E in* Ω, denoted by $P(E,\Omega)$, is defined by

$$P(E,\Omega) := \inf\left\{\liminf_{h\to+\infty} \int_\Omega g_h\, d\mu\colon\ (u_h) \subset \mathrm{Lip}_{\mathrm{loc}}(\Omega),\ u_h \xrightarrow{L^1_{\mathrm{loc}}(\Omega)} \chi_E,\ g_h \in D[u_h]\right\}.$$

We say that E has *finite perimeter in* Ω if $P(E,\Omega) < +\infty$.

Obviously, the same definitions hold in the case that $\Omega = X$. Notice, in particular, that $P(E,\Omega) = |D\chi_E|(\Omega)$.

We now state an Anzellotti–Giaquinta's density-type theorem (see [5]) for our setting. This result is an immediate consequence of Definition 2.2.

Theorem 2.3. *Let $\Omega \subset X$ be open and $u \in BV(\Omega)$. Then there exists a sequence $(u_h)_{h\in\mathbb{N}} \in \mathrm{Lip}_{\mathrm{loc}}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ and there exists $g_h \in D[u_h]$ such that*

$$u_h \xrightarrow{L^1_{\mathrm{loc}}(\Omega)} u, \qquad \int_\Omega g_h\, d\mu \to |Du|(\Omega) \quad \text{as } h \to \infty.$$

In the next proposition we collect some properties of the perimeter measure which can be found in [29] and [2].

Proposition 2.4. *Let $\Omega \subset X$ be open. Then the following properties hold*:

(i) (*Locality*) *$P(E,\Omega) = P(F,\Omega)$ whenever $(E \triangle F) \cap \Omega$ is μ-negligible, where $E \triangle F := (E \setminus F) \cup (F \setminus E)$*;
(ii) *(Lower semicontinuity) $E \to P(E,\Omega)$ is lower semicontinuous with respect to the $L^1_{\mathrm{loc}}(\Omega)$ topology*;
(iii) (*Subadditivity*) *$P(E \cup F,\Omega) + P(E \cap F,\Omega) \leqslant P(E,\Omega) + P(F,\Omega)$*;
(iv) (*Complementation*) *$P(E,\Omega) = P(E^c,\Omega)$.*

Theorem 2.5. *Let E be a set of finite perimeter in X. Then we have*:

(i) *The set function $P(E,\cdot)\colon A \to P(E,A)$ is the restriction to the open subsets of X of a finite regular Borel measure $P(E,\cdot)$ in X, defined by*
$$P(E,B) := \inf\bigl\{P(E,A)\colon\ A \supset B,\ A \text{ open}\bigr\} \quad \forall B \in \mathcal{B}(X). \tag{7}$$
(ii) *If D supports the weak* (1, 1)*-Poincaré inequality* (5)*, then the following weak* (1*, 1)*-Poincaré inequality holds*
$$\left(\frac{1}{\mu(B_r(x))}\int_{B_r(x)} \bigl|u(y) - u_{x,r}\bigr|^{s/(s-1)}\, d\mu(y)\right)^{(s-1)/s} \leqslant C\frac{r}{\mu(B_r(x))}\int_{B_{2\lambda r}(x)} g\, d\mu$$
for every $u \in \mathrm{Lip}(B_{2\lambda r}(x))$ and every $g \in D[u]$, where $s > 1$ is any exponent satisfying the density lower bound (4)*. Moreover, the following relative isoperimetric inequality holds*
$$\min\bigl\{\mu\bigl(E \cap B_r(x)\bigr), \mu\bigl(E^c \cap B_r(x)\bigr)\bigr\} \leqslant C_I\bigl[P\bigl(E, B_{2\lambda r}(x)\bigr)\bigr]^{s/(s-1)}, \tag{8}$$
where $C_I = C_I(s,x,r) = c(\frac{r^s}{\mu(B_r(x))})^{1/(s-1)}$ and $c = (2C)^{\frac{s}{s-1}}$.
(iii) *Let $\Omega \subset X$ be open and $u \in L^1_{\mathrm{loc}}(\Omega)$. Then we have the following* coarea formula:
$$|Du|(\Omega) = \int_{\mathbb{R}} P\bigl(\{u > t\},\Omega\bigr)\, dt. \tag{9}$$
Furthermore if $u \in BV(\Omega)$, then the set $\{x \in \Omega\colon\ u > t\}$ has finite perimeter for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$.

This theorem can be proved repeating the same arguments used in [29] (see also [2]).

Remark 2.6. We have the following basic facts:

(i) Let $E_1, E_2 \in \mathcal{B}(X)$ be such that $E_1 \cap \Omega = E_2 \cap \Omega$. Then

$$P(E_1, \Omega) = P(E_2, \Omega). \tag{10}$$

This follows from (i) of Proposition 2.4, since $(E_1 \triangle E_2) \cap \Omega = \emptyset$.

(ii) If $P(E, X) < \infty$, then

$$P(E, \Omega) + P\big(E, \Omega^c\big) = P(E, X). \tag{11}$$

This immediately follows from (i) of Theorem 2.5. We explicitly note that the quantity $P(E, \Omega^c)$ in (11) is defined by (7) of the previous Theorem 2.5.

We would stress that, using the set of axioms (A0)–(A5), the following connectivity property for the space X can be proved (see [2,12,25]):

Theorem 2.7 *(Quasi-convexity). For every $x, y \in X$, there exists a Lipschitz curve γ joining them with length at most $Cd(x, y)$. Here C is a constant depending only on the constants C_D, λ and C_P which appear in the axioms* (A0)–(A5).

Finally, we recall the following useful consequence of the doubling condition whose proof can be found in [35].

Proposition 2.8 *(Reverse doubling condition). There exist $\alpha, \beta > 1$ such that*

$$\mu\big(B_{\alpha\rho}(x)\big) \geqslant \beta\mu\big(B_\rho(x)\big) \quad \forall x \in X,\ \rho > 0.$$

Proof. We just have to notice that in [35] the metric space X satisfies the further hypothesis that every annulus $B(x, R) \setminus B(x, r)$ $(x \in X,\ 0 < r < R < +\infty)$ is nonempty. This condition is easily seen to hold in our setting, for instance, by applying Theorem 2.7. □

Remark 2.9. Let $\bar{x} \in X$ and $\delta \in]0, 1]$. By a quite elementary argument one can show that the reverse doubling condition implies the following inequality:

$$\mu\big(B_{\delta\rho}(\bar{x})\big) \leqslant \beta\delta^{\log_\alpha \beta}\mu\big(B_\rho(\bar{x})\big). \tag{12}$$

We also stress that from (12) it follows immediately that $\mu(X) = +\infty$ if we assume that the *diameter* $\operatorname{diam}(X)$ of the ambient space X is not finite, where we have set $\operatorname{diam}(X) := \sup\{d(x, y)\colon\ x, y \in X,\ x \neq y\}$. Indeed, to this aim, put $\delta = 1/\rho$ in formula (12) and let ρ to $+\infty$. More generally, it can be showed that the measure of any doubling metric space is finite if and only if its diameter is finite; see also Remark 2.3 in [3].

3. Main results: Extension of *BV* functions

In the sequel $\Omega \subset X$ will denote an open set. Moreover, we recall that, for any $B \in \mathcal{B}(X)$, $P(B, \Omega^c)$ is defined by (7) of Theorem 2.5.

Definition 3.1. Let $\Omega \subset X$ be open. To any $E \subset \Omega$, $E \in \mathcal{B}(X)$, we associate the set function $\tau(\cdot, \Omega)\colon E \to \tau(E, \Omega)$ defined by

$$\tau(E, \Omega) := \inf_{B \cap \Omega = E} P\big(B, \Omega^c\big) \quad \forall B \in \mathcal{B}(X).$$

Remark 3.2. We stress that $\tau(E, \Omega) = \tau(\Omega \setminus E, \Omega)$. Indeed, by applying (iv) of Proposition 2.4 to any open set $A \supset \Omega^c$ and then by using (7) of Theorem 2.5, we get that $P(B, \Omega^c) = P(B^c, \Omega^c)$ for every $B \in \mathcal{B}(X)$. Hence, the claim follows by noting that $B \cap \Omega = E$ if and only if $B^c \cap \Omega = \Omega \setminus E$.

Now we are in a position to state our main result connecting the existence of an extension to the whole space X for a $BV(\Omega)$ function to an "isoperimetric-type inequality" involving the set function $\tau(\cdot, \Omega)$ associated with Ω.

We point out that our proof follows basically the lines of the original one [10] (see also §6.3.3 of [28]) and so in the sequel we will try to stress the differences due to our abstract setting.

Theorem 3.3. *Let $\Omega \subset X$ be open. If for any function $u \in BV(\Omega)$ there exists an extension $\hat{u} \in BV(X)$ such that*

$$|D\hat{u}|(X) \leqslant C|Du|(\Omega), \tag{13}$$

where C is a constant independent of u, then

$$\tau(E,\Omega) \leqslant (C-1)P(E,\Omega) \tag{14}$$

for all Borel sets $E \subset \Omega$. Conversely, if for all Borel sets $E \subset \Omega$ inequality (14) *is true with a constant C that is independent of E, then for any $u \in BV(\Omega)$ there exists an extension $\hat{u} \in BV(X)$ such that* (13) *holds.*

Proof of the necessity of condition (14). Inequality (14) is trivial if $P(E,\Omega) = +\infty$. So let us assume that $P(E,\Omega)$ is finite. By hypothesis there exists an extension $\hat{\chi}_E$ of the characteristic function χ_E such that $|D\hat{\chi}_E|(X) \leqslant CP(E,\Omega)$. This fact, together with the coarea formula (9) of Theorem 2.5, implies the following chain of inequalities:

$$CP(E,\Omega) \geqslant |D\hat{\chi}_E|(X) \geqslant \int_{\mathbb{R}} P\big(\{\hat{\chi}_E > t\}, X\big)\,dt \geqslant \int_0^1 P\big(\{\hat{\chi}_E > t\}, X\big)\,dt.$$

Since for every $t \in]0,1[$ one has $\{\hat{\chi}_E > t\} \cap \Omega = E$, we have $P(\{\hat{\chi}_E > t\}, X) \geqslant \inf_{B\cap\Omega=E} P(B,X)$ where $B \in \mathcal{B}(X)$, and hence $C\,P(E,\Omega) \geqslant \inf_{B\cap\Omega=E} P(B,X)$. By using (11) and (10) we get

$$CP(E,\Omega) \geqslant \inf_{B\cap\Omega=E} P(B,X) \geqslant \inf_{B\cap\Omega=E} P(B,\Omega) + \inf_{B\cap\Omega=E} P\big(B,\Omega^c\big) \geqslant P(E,\Omega) + \tau(E,\Omega). \qquad \square$$

In order to prove the sufficiency of condition (14), we have to state some preliminary lemmas.

Lemma 3.4. *If $N \subset \Omega$, $\tau(N,\Omega) < \infty$ and $P(N,\Omega) < \infty$, then there exists a set $E \subset X$ such that $E \cap \Omega = N$ and*

$$P\big(E,\Omega^c\big) = \tau(N,\Omega). \tag{15}$$

Proof. The proof of this fact is based on the same argument as the Euclidean one (see Lemma 1 in [10]). However, for the reader's convenience, we report it below.

Let $\{E_i\}_{i\in\mathbb{N}} \subset X$ be a sequence of subsets of X such that $E_i \cap \Omega = N$ and $\lim_{i\to\infty} P(E_i,\Omega^c) = \tau(N,\Omega)$. In particular,

$$\sup_{i\in\mathbb{N}} P\big(E_i,\Omega^c\big) < \infty.$$

Now since, by locality, $P(E_i,\Omega) = P(N,\Omega)$, we get that $\sup_{i\in\mathbb{N}} P(E_i,X) < \infty$. Thus, by the compact embedding of the space $BV_{\mathrm{loc}}(X)$ in $L^1_{\mathrm{loc}}(X)$ (see Theorem 3.7 in [29]) there exists a subsequence, still denoted by $\{E_i\}$, converging to some subset E of X. By lower semicontinuity of the perimeter measure we have

$$P(E,X) \leqslant \liminf_{i\to\infty} P(E_i,X).$$

Moreover, since $E \cap \Omega = N$ and $P(E_i,\Omega) = P(N,\Omega)$, we get

$$P(E,\Omega) + P\big(E,\Omega^c\big) \leqslant P(N,\Omega) + \lim_{i\to\infty} P\big(E_i,\Omega^c\big) = P(E,\Omega) + \tau(N,\Omega).$$

Hence $P(E,\Omega^c) \leqslant \tau(N,\Omega)$ and from very definition of $\tau(N,\Omega)$ we get (15). $\square$

In Proposition 2.4 the subadditivity property of the perimeter measure with respect to an open set $\Omega \subset X$ was stated. We need now to extend this property to the case of the closed set Ω^c.

Lemma 3.5. *If $E_1, E_2 \in \mathcal{B}(X)$, then we have*

$$P\big(E_1 \cap E_2,\Omega^c\big) + P\big(E_1 \cup E_2,\Omega^c\big) \leqslant P\big(E_1,\Omega^c\big) + P\big(E_2,\Omega^c\big). \tag{16}$$

Proof. For any $\varepsilon > 0$, by using the very definition of the perimeter measure for Borel sets (see (i) of Theorem 2.5), it follows that there exist open sets U_ε, V_ε such that $U_\varepsilon \supset \Omega^c$, $V_\varepsilon \supset \Omega^c$ and for which $P(E_1, \Omega^c) \geqslant P(E_1, U_\varepsilon) - \varepsilon$ and $P(E_2, \Omega^c) \geqslant P(E_2, V_\varepsilon) - \varepsilon$. By monotonicity of the perimeter measure and (iii) of Proposition 2.4, we get

$$\begin{aligned} P\big(E_1 \cap E_2, \Omega^c\big) + P\big(E_1 \cup E_2, \Omega^c\big) &\leqslant P(E_1 \cap E_2, U_\varepsilon \cap V_\varepsilon) + P(E_1 \cup E_2, U_\varepsilon \cap V_\varepsilon) \\ &\leqslant P(E_1, U_\varepsilon \cap V_\varepsilon) + P(E_2, U_\varepsilon \cap V_\varepsilon) \leqslant P(E_1, U_\varepsilon) + P(E_2, V_\varepsilon) \\ &\leqslant P\big(E_1, \Omega^c\big) + P\big(E_2, \Omega^c\big) + 2\varepsilon \end{aligned}$$

and (16) follows. □

Lemma 3.6. *Let $E_1, E_2 \in \mathcal{B}(X)$ be such that $P(E_k, \Omega^c) < \infty$ $(k = 1, 2)$. We set $B_k := E_k \cap \Omega$ $(k = 1, 2)$. If $B_1 \subset B_2$ and $P(E_k, \Omega^c) = \tau(B_k, \Omega)$ $(k = 1, 2)$, then*

$$P\big(E_1 \cap E_2, \Omega^c\big) = P\big(E_1, \Omega^c\big), \qquad P\big(E_1 \cup E_2, \Omega^c\big) = P\big(E_2, \Omega^c\big).$$

Proof. We stress that this proof mimics the corresponding Euclidean one (see Lemma 3 in [10]).

Since $E_1 \cap E_2 \cap \Omega = B_1$ and $(E_1 \cup E_2) \cap \Omega = B_2$, by the very definition of $\tau(\cdot, \Omega)$ we get that

$$\tau(B_1, \Omega) \leqslant P\big(E_1 \cap E_2, \Omega^c\big), \qquad \tau(B_2, \Omega) \leqslant P\big(E_1 \cup E_2, \Omega^c\big). \tag{17}$$

By hypothesis, $P(E_k, \Omega^c) = \tau(B_k, \Omega)$ $(k = 1, 2)$ and so using (16) yields

$$P\big(E_1 \cap E_2, \Omega^c\big) + P\big(E_1 \cup E_2, \Omega^c\big) \leqslant \tau(B_1, \Omega) + \tau(B_2, \Omega),$$

which together with (17) proves the lemma. □

In the next lemma we prove an isoperimetric-type inequality.

Lemma 3.7. *Let $B_r(x) \subset X$ be the open ball of radius r centred at the fixed point $x \in X$ and let $\varepsilon > 0$. Then we have that*

$$\mu(E)^{(s-1)/s} \leqslant C(r, \varepsilon) P\big(E, B_{2\lambda r}(x)\big) \tag{18}$$

holds for any Borel set E contained in $B_r(x)$ such that $\mu(E) \leqslant (1 - \varepsilon)\mu(B_r(x))$, where s is any exponent satisfying the density lower bound (4) *and $\lambda \geqslant 1$.*

Proof. First notice that if E is not of finite perimeter in $B_{2\lambda r}(x)$ then (18) is trivial. So let us assume that E has finite perimeter in $B_{2\lambda r}(x)$. Moreover, let us remark that by means of Theorem 2.3 we can prove that the weak $(1^*, 1)$-Poincaré inequality (see (ii) of Theorem 2.5) holds for every function $u \in BV(B_{2\lambda r}(x))$. So we can apply the weak $(1^*, 1)$-Poincaré inequality with $u = \chi_E$ and we easily get that

$$\varepsilon\mu(E)^{(s-1)/s} \leqslant \bigg(\int_{B_r} \bigg|\chi_E - \frac{\mu(E)}{\mu(B_r(x))}\bigg|^{s/(s-1)} d\mu\bigg)^{(s-1)/s} \leqslant C\bigg(\frac{r\mu(B_r(x))^{(s-1)/s}}{\mu(B_r(x))}\bigg) P\big(E, B_{2\lambda r}(x)\big). \qquad \square$$

Proof of the sufficiency of condition (14). First, let us set

$$N_t := \big\{x \in \Omega\colon\ u(x) \geqslant t\big\}.$$

We claim that there exists a family of sets B_t such that:

$$(1)\quad B_t \cap \Omega = N_t, \qquad (2)\quad P\big(B_t, \Omega^c\big) = \tau(N_t, \Omega), \qquad (3)\quad B_t \subset B_\tau \quad \text{for } t > \tau.$$

Below, we shall divide the proof of the existence of such a family into three steps. In the first step we will construct B_t for a countable family of numbers $\{t_i\}_{i \in \mathbb{N}}$ which is dense in $\mathbb{R}$ and then, in the second step, we will prove that this fact holds for every $t \in \mathbb{R}$. Finally, in the third step, we will introduce a function $\hat{u}(x)$ defined as $\hat{u}(x) := \sup\{t \in \mathbb{R}\colon\ x \in B_t\}$ and show that $\hat{u}$ satisfies (13).

Step 1. Let $u \in BV(\Omega)$. Moreover, let N_t be as above. Since

$$\chi_{\{x:\ u(x) \geqslant t\}} = L^1 - \lim_{k\to\infty} \chi_{\{x:\ u(x) > t - \frac{1}{k}\}} \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \mathbb{R},$$

by lower semicontinuity of the perimeter and by Fatou's Lemma we get

$$\int_{\mathbb{R}} P\big(\{x:\ u(x) \geqslant t\}, \Omega\big)\, dt \leqslant \int_{\mathbb{R}} \liminf_{k\to\infty} P\left(\left\{x:\ u(x) > t - \frac{1}{k}\right\}, \Omega\right) dt.$$

Making the change of variable $s := t - \frac{1}{k}$, by the coarea formula (9) we get

$$\int_{\mathbb{R}} P\big(\{x:\ u(x) \geqslant t\}, \Omega\big)\, dt \leqslant \int_{\mathbb{R}} P\big(\{x:\ u(x) > s\}, \Omega\big)\, ds = |Du|(\Omega).$$

In particular, $P(N_t, \Omega) < \infty$ for $\mathcal{L}^1$- a.e. $t \in \mathbb{R}$. Thus, we can choose a countable set $\{t_i:\ i \in \mathbb{N}\}$, $t_i \neq t_j$ if $i \neq j$, which is everywhere dense in $\mathbb{R}$ and such that $P(N_{t_i}, \Omega) < \infty$. The hypothesis (14) implies that $\tau(N_{t_i}, \Omega) < \infty$. Then, we may construct a sequence of sets $\{B_{t_i}\}_{i\in\mathbb{N}}$ such that:

$$(1)\quad B_{t_i} \cap \Omega = N_{t_i}, \qquad (2)\quad P\big(B_{t_i}, \Omega^c\big) = \tau(N_{t_i}, \Omega), \qquad (3)\quad B_{t_i} \subset B_{t_j} \quad \text{for } t_i > t_j.$$

The proof of the last claim can be carried out with no substantial changes from [10], by making use of Lemmas 3.4 and 3.6.

Step 2. If $t \notin \{t_i\}$, from the set $\{t_i\}$ we extract two monotone sequences $\{a_i\}$ and $\{b_i\}$ such that $a_i < t < b_i$ and $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = t$. Since $P(N_t, \Omega) < \infty$ and $\tau(N_t, \Omega) < \infty$, according to Lemma 3.4, there exists a set $B_t^{(0)}$ such that $B_t^{(0)} \cap \Omega = N_t$ and $P(B_t^{(0)}, \Omega^c) = \tau(N_t, \Omega)$. We consider the sequence of sets $B_t^{(k)} = B_t^{(0)} \cap B_{a_k}$ $(k = 1, 2, \ldots)$. Since $t > a_k$ for $x \in N_t$, then $x \in N_{a_k}$ for all k and so $B_t^{(k)} \cap \Omega = (B_t^{(0)} \cap \Omega) \cap (B_{a_k} \cap \Omega) = N_t$. Obviously $B_t^{(k+1)} \subset B_t^{(k)}$. Applying Lemma 3.6 we get $P(B_t^{(k)}, \Omega^c) = \tau(N_t, \Omega)$ for every $k = 1, 2, \ldots$. We now introduce the set $\tilde{B}_t := \bigcap_{k=1}^{\infty} B_t^{(k)}$. We have $P(\tilde{B}_t, \Omega) < \infty$ and $P(\tilde{B}_t, \Omega^c) < \infty$ and hence $P(\tilde{B}_t, X) < \infty$. Since $B_t^{(k)} \to \tilde{B}_t$ as $k \to \infty$, then by lower semicontinuity of the perimeter measure we get $P(\tilde{B}_t, X) \leqslant \liminf_{k\to\infty} P(B_t^{(k)}, X)$.

Now we have to prove that the same inequality holds for Ω^c. To this end we make use of the decomposition formula (11) and of the locality property of the perimeter (see (i) in Proposition 2.4) which gives us $P(B_t^{(k)}, \Omega) = P(\tilde{B}_t, \Omega)$. So we have:

$$\begin{aligned} P\big(\tilde{B}_t, \Omega^c\big) &= P(\tilde{B}_t, X) - P(\tilde{B}_t, \Omega) \leqslant \liminf_{k\to\infty}\big(P\big(B_t^{(k)}, X\big) - P(\tilde{B}_t, \Omega)\big) \\ &= \liminf_{k\to\infty}\big(P\big(B_t^{(k)}, X\big) - P\big(B_t^{(k)}, \Omega\big)\big) = \liminf_{k\to\infty} P\big(B_t^{(k)}, \Omega^c\big) = \tau(N_t, \Omega). \end{aligned}$$

To obtain the converse inequality it suffices to recall that $\tilde{B}_t \cap \Omega = N_t$. Thus $P(\tilde{B}_t, \Omega^c) = \tau(N_t, \Omega)$. The rest of the proof is essentially the same as the corresponding one in [10]. In particular, we may construct Borel sets B_t which satisfy the conditions:

$$(1)\quad B_t \cap \Omega = N_t, \qquad (2)\quad P\big(B_t, \Omega^c\big) = \tau(N_t, \Omega), \qquad (3)\quad B_\tau \subset B_t \quad \text{for } t < \tau.$$

Step 3. Following [10] (see Item 4, p. 48), let us set $\hat{u}(x) := \sup\{t:\ x \in B_t\}$. In order to show that $\hat{u}$ is locally summable one needs the isoperimetric-type inequality (18) stated in Lemma 3.7. To be more precise, let $\overline{B}_\delta$ be any closed ball contained in Ω and let B_r be another ball such that $B_r \supset \overline{B}_\delta$. By making use of (18) applied to $\mu(E \setminus B_\delta)$, where $E \subset B_r$ is any Borel set, we get

$$\begin{aligned} \mu(E) &= \mu(E \cap B_\delta) + \mu(E \setminus B_\delta) \leqslant \mu(E \cap B_\delta) + C(r, \delta) P(E, B_{2\lambda r})^{s/(s-1)} \\ &\leqslant \mu(E \cap B_\delta) + \tilde{C}(r, \delta) P(E, B_{2\lambda r}) \leqslant \widehat{C}(r, \delta)\big(\mu(E \cap B_\delta) + P(E, X)\big). \end{aligned}$$

Now let us set $E := B_t \cap B_r$ for $t \geqslant 0$ and $E := B_r \setminus B_t$ for $t < 0$. By substituting these expressions in the latter inequality, using the fact that $P(B_t, \Omega^c) = \tau(N_t, \Omega)$ and inequality (14), we can argue again as in [10] and, explicitly, we get that

$$\int_{B_r} |\hat{u}|\, d\mu \leqslant \widehat{C}(r,\delta)\biggl(C|Du|(\Omega) + \int_{B_\delta} |u|\, d\mu\biggr),$$

which implies the local summability of $\hat{u}$. Finally, as in [10], we may prove that $\hat{u} \in BV(X)$ and that inequality (13) holds true by applying the coarea formula (9) and the inequality (14). The proof of the theorem is complete. □

We have to emphasize that the "geometric condition" (14) required in the above Theorem 3.3 is global, in the sense that, for instance, it is not satisfied if Ω is not connected. However, this difficulty can easily be overcome by requiring a weaker condition. The next result generalizes Theorem 2 in [10]. Nevertheless, we stress that the proof of the necessity condition stated below requires a different argument with respect to the Euclidean one where the proof is shortly achieved by means of the global isoperimetric inequality. Actually, in our proof we shall make use of the relative isoperimetric inequality and of the reverse doubling condition (see Proposition 2.8), since in our metric setting a global isoperimetric inequality does not make sense, in general.

Theorem 3.8. *Let $\Omega \subset X$ be a bounded open set. Then the following conditions are equivalent*:

(i) *for any function $u \in BV(\Omega)$ there exists an extension $\hat{u} \in BV(X)$ satisfying*

$$|D\hat{u}|(X) \leqslant K_1 \|u\|_{BV(\Omega)}, \tag{19}$$

where K_1 is independent of u;

(ii) *there exists $\delta > 0$ such that for every Borel set $E \subset \Omega$ satisfying* $\operatorname{diam}(E) \leqslant \delta$ *one has*

$$\tau(E,\Omega) \leqslant K_2 P(E,\Omega), \tag{20}$$

where K_2 is independent of E.

Proof. (i) ⇒ (ii). Let $E \subset \Omega$ be a set of finite perimeter in Ω and let $\hat{\chi}_E$ denote the extension of χ_E satisfying (19). Then

$$I := K_1\bigl(P(E,\Omega) + \mu(E)\bigr) \geqslant |D\hat{\chi}_E|(X) \geqslant \int_0^1 P\bigl(\{x \in X\colon\ \hat{\chi}_E \geqslant t\}, X\bigr)\, dt.$$

Since $E = \{x \in X\colon\ \hat{\chi}_E \geqslant t\} \cap \Omega$ whenever $t \in\,]0,1[$, we also get

$$I \geqslant \inf_{B \cap \Omega = E} P(B,X), \tag{21}$$

where $B \in \mathcal{B}(X)$ is any set of finite perimeter in X.

Step 1. We have

$$I \geqslant C_I^{\frac{1-s}{s}} \min\bigl\{\bigl(\mu(E)\bigr)^{\frac{s-1}{s}}, \bigl(\mu(\Omega \setminus E)\bigr)^{\frac{s-1}{s}}\bigr\}. \tag{22}$$

To prove this assertion, let us fix $\bar{x} \in E$ and let $\rho > 0$ be such that $\Omega \subset B_\rho(\bar{x})$. Moreover, let B be a set of finite perimeter in X such that $B \cap \Omega = E$. Clearly we have that $E \subset B \cap B_\rho(\bar{x})$ and that $(\Omega \setminus E) \subset B^c \cap B_\rho(\bar{x})$. By applying the relative isoperimetric inequality (8) to the set B and to the ball $B_\rho(\bar{x})$, we get

$$II := \min\bigl\{\mu\bigl(B \cap B_\rho(\bar{x})\bigr), \mu\bigl(B^c \cap B_\rho(\bar{x})\bigr)\bigr\} \leqslant C_I \bigl[P\bigl(B, B_{2\lambda\rho}(\bar{x})\bigr)\bigr]^{s/(s-1)} \leqslant C_I \bigl[P(B,X)\bigr]^{s/(s-1)}. \tag{23}$$

So if $II = \mu(B \cap B_\rho(\bar{x}))$, we get that $\mu(E) \leqslant \mu(B \cap B_\rho(\bar{x}))$. On the other hand, if $II = \mu(B^c \cap B_\rho(\bar{x}))$, we get that $\mu(\Omega \setminus E) \leqslant \mu(B^c \cap B_\rho(\bar{x}))$. Thus, in any case, (22) follows by using (23) and (21).

Step 2. As we know from Theorem 2.5, the isoperimetric constant C_I is given by $C_I = C_I(s, \bar{x}, \rho) = c(\frac{\rho^s}{\mu(B_\rho(\bar{x}))})^{\frac{1}{s-1}}$. Since $\mu(B_\rho(\bar{x})) \geqslant \min_{x\in\overline{\Omega}} \mu(B_\rho(x))$ (here $\overline{\Omega}$ denotes the closure of the set Ω) we get

$$C_I^* := c\left(\frac{\rho^s}{\min_{x\in\overline{\Omega}} \mu(B_\rho(x))}\right)^{1/(s-1)} \geqslant C_I,$$

where we stress that C_I^* turns out to be independent of the set E. By arguing exactly as in Step 1 we therefore get

$$I \geqslant C_I^{*\frac{1-s}{s}} \min\{(\mu(E))^{\frac{s-1}{s}}, (\mu(\Omega \setminus E))^{\frac{s-1}{s}}\}. \tag{24}$$

Step 3. We claim that: *there exists $\delta > 0$ such that $\mu(E) \leqslant P(E, \Omega)$ for every Borel set $E \subset \Omega$ such that* $\operatorname{diam}(E) < \delta$.

Let us fix a point $\bar{x} \in E$ and let $\rho > 0$ be such that $\Omega \subset B_\rho(\bar{x})$. Without loss of generality, we may assume that $B_{\operatorname{diam}(E)}(\bar{x}) \subset B_\rho(\bar{x})$. So we have $\mu(E) \leqslant \mu(B_{\operatorname{diam}(E)}(\bar{x})) \leqslant \mu(B_\rho(\bar{x}))$. By the reverse doubling condition (see Remark 2.9), there exist $\alpha, \beta > 1$ such that

$$\mu(B_{\operatorname{diam}(E)}(\bar{x})) \leqslant \beta\left(\frac{\operatorname{diam}(E)}{\rho}\right)^{\log_\alpha \beta} \mu(B_\rho(\bar{x})). \tag{25}$$

Set now

$$\delta_0 := \rho\left(\frac{\mu(\Omega)}{2\beta \max_{x\in\overline{\Omega}} \mu(B_\rho(x))}\right)^{\frac{1}{\log_\alpha \beta}}.$$

If $\operatorname{diam}(E) \leqslant \delta_0$, from (25) it follows that $\mu(E) \leqslant \frac{1}{2}\mu(\Omega)$, and therefore that

$$\mu(E) \leqslant \mu(\Omega \setminus E). \tag{26}$$

At this point, let us set $\gamma := \frac{C_I^{*\frac{1-s}{s}}}{K_1}$. From Step 2 it follows that the constant γ does not depends on E. Moreover, by means of the inequality (21), we can conclude Step 3 by showing that there exists $\delta \leqslant \delta_0$ such that

$$\mu(E)^{-\frac{1}{s}} \geqslant \frac{2}{\gamma} \tag{27}$$

for every Borel set E such that $\operatorname{diam}(E) < \delta$. Indeed, by (27), and by using (24) and (26), we obtain

$$P(E, \Omega) + \mu(E) \geqslant \gamma(\mu(E))^{\frac{s-1}{s}} \geqslant 2\mu(E),$$

that is what we claim in Step 3. Now we prove (27). Consider again condition (25); then (27) follows by choosing $\delta \leqslant \min\{\delta_0, \delta_1\}$, where we have set

$$\delta_1 := \rho\left(\frac{1}{\beta \max_{x\in\overline{\Omega}} \mu(B_\rho(x))}\left(\frac{\gamma}{2}\right)^s\right)^{\frac{1}{\log_\alpha \beta}}.$$

Finally, to achieve the proof of the first part of the theorem we just have to notice that by inequality (21), by Step 3, together with the very definition of $\tau(\cdot, \Omega)$, it follows that

$$2K_1 P(E, \Omega) \geqslant \inf_{B\cap\Omega=E} P(B, X) \geqslant \tau(E, \Omega).$$

(ii) $\Rightarrow$ (i). The proof of this implication proceeds similarly as the original one of [10]. The main ingredient that we have to adapt to our setting is a partition of unity. So let us choose a covering $\{B_i^*\}$ of Ω by balls of radius δ and with the property that $\frac{1}{5}B_i^*$ are pairwise disjoint. We now set $B_i = 2B_i^*$. By the doubling condition it is easy to check that there exists a number n such that no point of $\overline{\Omega}$ belongs to more than n balls B_i^*. Let $\{\psi_i\}$ be a Lipschitz partition of unity associated to $\{B_i\}$. This means that $\psi_i \in \mathrm{Lip}_{\mathrm{loc}}(X)$, $\sum_i \psi_i = 1$, $0 \leqslant \psi_i \leqslant 1$, $\operatorname{supp}\psi_i \subset B_i$ and all functions can be chosen with the same Lipschitz constant $L = n/\delta$. An explicit construction of a family of functions ψ_i satisfying these requirements can be found, for example, in [16]. Since Ω is bounded, without loss of generality, we may assume that the covering is finite and that m is the cardinality of $\{B_i\}$. Now let $u \in BV(\Omega)$ and set $u_i := \psi_i u$.

Using Definition 2.2, Leibniz rule (see axiom (A3)) and the properties of ψ one can easily check that $u_i \in BV(\Omega)$ and that

$$|Du_i|(\Omega) \leqslant |Du|(\Omega) + L\|u\|_{L^1(\Omega)}.$$

Set now $N_t := \{x \in X\colon u_i(x) \geqslant t\}$. Clearly, for every $t \neq 0$ we have $\operatorname{diam} N_t < \delta$ and so

$$\tau(N_t, \Omega) \leqslant K_2 P(N_t, \Omega).$$

At this point we may apply the same proof as the one of the sufficiency of condition (14) of Theorem 3.3. So we obtain the existence of a finite family of functions $\{\hat{u}_i\}$ such that $\hat{u}_i \in BV(X)$, $\hat{u}_i|_\Omega = u_i$ and satisfying

$$|D\hat{u}_i|(X) \leqslant (K_2+1)|Du_i|(\Omega) \leqslant (K_2+1)L\|u\|_{BV(\Omega)}.$$

Finally, let $\hat{u} := \sum_{i=1}^m \hat{u}_i$. It is immediate to see that $\hat{u}|_\Omega = u$ and that

$$|D\hat{u}|(X) \leqslant m(K_2+1)|Du_i|(\Omega) \leqslant (K_2+1)L\|u\|_{BV(\Omega)}$$

which is the thesis. □

Finally, we could also restate Theorem 3.3 in terms of an extension linear operator denoted as $\mathcal{E}_\Omega : BV(\Omega) \to BV(X)$ which associates to every function $u \in BV(\Omega)$ its extension $\hat{u} \in BV(X)$. More precisely, we have the following

Corollary 3.9. *If $\Omega \subset X$ is open, we set*

$$|\Omega| := \inf\bigl\{k \in \mathbb{R}\colon \tau(E, \Omega) \leqslant kP(E, \Omega)\ \forall E \subset \Omega\bigr\}.$$

Then there exists a bounded linear operator $\mathcal{E}_\Omega : BV(\Omega) \to BV(X)$ if and only if $|\Omega|$ is finite. Moreover, for any extension operator $\mathcal{E}_\Omega$ one has

$$1 + |\Omega| \leqslant \|\mathcal{E}_\Omega\|$$

and, in particular, there exists an extension operator $\widetilde{\mathcal{E}_\Omega}$ satisfying $\|\widetilde{\mathcal{E}_\Omega}\| = 1 + |\Omega|$. Here $\|\cdot\|$ denotes the usual norm of a linear operator.

Proof. It is just a restatement of Theorem 3.3. □

Acknowledgments

Both authors would like to thank Bruno Franchi for very useful conversations and his interest in this work. Many thanks are due to the referees for their useful suggestions.

References

[1] L. Ambrosio, Some fine properties of sets of finite perimeter in Ahlfors regular metric spaces, Adv. Math. 159 (1) (2001) 51–67.
[2] L. Ambrosio, Fine properties of sets of finite perimeter in doubling metric spaces, Set-Valued Anal. 10 (2–3) (2002) 111–128.
[3] L. Ambrosio, M. Miranda Jr., D. Pallara, Special Functions of Bounded Variation in Doubling Metric Measure Spaces, in: Calculus of Variations: Topics from the Mathematical Heritage of Ennio De Giorgi, in: Quad. Mat., vol. 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 1–45.
[4] L. Ambrosio, P. Tilli, Topics on Analysis in Metric Spaces, Oxford Lecture Ser. Math. Appl., vol. 25, 2004.
[5] G. Anzellotti, M. Giaquinta, Funzioni BV e tracce, Rend. Sem. Mat. Univ. Padova 60 (1978) 1–21 (1979).
[6] A. Baldi, Weighted *BV* functions, Houston J. Math. 27 (3) (2001) 683–705.
[7] A. Baldi, B. Franchi, A Γ-convergence result for doubling metric measures and associated perimeters, Calc. Var. Partial Differential Equations 16 (3) (2003) 283–298.
[8] A. Baldi, B. Franchi, Mumford–Shah type functionals associated with doubling metric measures, Proc. Roy. Soc. Edinburgh Sect. A 135 (1) (2005) 1–23.
[9] A. Björn, J. Björn, N. Shanmugalingam, Sobolev extensions of Hölder continuous and characteristic functions on metric spaces, Canad. J. Math., in press.
[10] Yu.D. Burago, V.G. Maz'ja, Potential Theory and Function Theory for Irregular Regions, Seminars in Math., vol. 3, V.A. Steklov Math. Inst., Leningrad, 1969 (translated from Russian).
[11] L. Capogna, D. Danielli, N. Garofalo, The geometric Sobolev embedding for vector fields and the isoperimetric inequality, Comm. Anal. Geom. 2 (2) (1994) 203–215.

[12] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. 9 (3) (1999) 428–517.
[13] S.K. Chua, Some remarks on extension theorems for weighted Sobolev spaces, Illinois J. Math. 38 (1) (1994) 95–126.
[14] D. Danielli, N. Garofalo, D.M. Nhieu, Non-doubling Ahlfors measures, perimeter measures, and the characterization of the trace spaces of Sobolev functions in Carnot Carathéodory spaces, Mem. Amer. Math. Soc. 182 (857) (2006) x+119 pp.
[15] G. David, S. Semmes, Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings, in: Analysis and Partial Differential Equations, in: Lecture Notes in Pure and Appl. Math., vol. 122, Dekker, New York, 1990.
[16] B. Franchi, P. Hajłasz, P. Koskela, Definitions of Sobolev classes on metric spaces, Ann. Inst. Fourier (Grenoble) 49 (6) (1999) 1903–1924.
[17] B. Franchi, C. Pérez, R.L. Wheeden, Self-improving properties of John–Nirenberg and Poincaré inequalities on spaces of homogeneous type, J. Funct. Anal. 153 (1) (1998) 108–146.
[18] B. Franchi, C. Pérez, R.L. Wheeden, A sum operator with applications to self-improving properties of Poincaré inequalities in metric spaces, J. Fourier Anal. Appl. 9 (5) (2003) 511–540.
[19] B. Franchi, R. Serapioni, F. Serra Cassano, Meyers–Serrin type theorems and relaxation of variational integrals depending on vector fields, Houston J. Math. 22 (4) (1996) 859–889.
[20] V. Gol'dshtein, M. Troyanov, Axiomatic theory of Sobolev spaces, Exp. Math. 19 (4) (2001) 289–336.
[21] A.V. Greshnov, On uniform and NTA-domains on Carnot groups, Sibirsk. Mat. Zh. 42 (5) (2001) 1018–1035 (in Russian); translation in Siberian Math. J. 42 (5) (2001) 851–864.
[22] A. Grigor'yan, Isoperimetric inequalities and capacities on Riemannian manifolds, in: The Maz'ya Anniversary Collection, vol. 1, Rostock, 1998, in: Oper. Theory Adv. Appl., vol. 109, Birkhäuser, Basel, 1999, pp. 139–153.
[23] M. Gromov, Metric Structures for Riemannian and Non-Riemannian Spaces, Progr. Math., vol. 153, Birkhäuser, Boston, 1999.
[24] P. Hajłasz, Sobolev spaces on an arbitrary metric space, Potential Anal. 5 (1996) 403–415.
[25] P. Hajłasz, P. Koskela, Sobolev met Poincaré, Mem. Amer. Math. Soc. 145 (688) (2000) x+101 pp.
[26] P. Hajłasz, P. Koskela, H. Tuominen, Measure density and extendability of Sobolev functions, preprint.
[27] J. Heinonen, Lectures on Analysis on Metric Spaces, Universitext, Springer-Verlag, New York, 2001.
[28] V.G. Maz'ja, Sobolev Spaces, Springer-Verlag, 1985.
[29] M. Miranda Jr., Functions of bounded variation on "good" metric measure spaces, J. Math. Pures Appl. (9) 82 (8) (2003) 975–1004.
[30] R. Monti, D. Morbidelli, Regular domains in homogeneous groups, Trans. Amer. Math. Soc. 357 (8) (2005) 2975–3011.
[31] R. Monti, D. Morbidelli, Non-tangentially accessible domains for vector fields, Indiana Univ. Math. J. 54 (2) (2005) 473–498.
[32] S. Semmes, Metric spaces and mappings seen at many scales, in: M. Gromov (Ed.), Metric Structures for Riemannian and non-Riemannian Spaces. With Appendices by M. Katz, P. Pansu, S. Semmes, in: J. LaFontaine, P. Pansu (Eds.), Progr. Math., vol. 152, Birkhäuser, Boston, 1999.
[33] N. Shanmugalingam, Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, Rev. Mat. Iberoamericana 16 (2000) 243–279.
[34] P. Shvartsman, On extensions of Sobolev functions defined on regular subsets of metric measure spaces, J. Approx. Theory 144 (2007) 139–161.
[35] R.L. Wheeden, A characterization of some weighted norm inequalities for the fractional maximal function, Studia Math. 107 (3) (1993) 257–272.